Wormholes with a barotropic equation of state admitting a one-parameter group of conformal motions

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Highlights
The theoretical construction of Morris-Thorne wormholes retains complete control over the geometry at the expense of the stress-energy tensor.
The introduction of a barotropic equation of state fails to produce a solution even if the energy density is known.
The assumption of conformal symmetry fills the gap in the form of a complete wormhole solution.

Abstract

The theoretical construction of a traversable wormhole proposed by Morris and Thorne maintains complete control over the geometry by assigning both the shape and redshift functions, thereby leaving open the determination of the stress-energy tensor. This paper examines the effect of introducing the linear barotropic equation of state $p_r = \omega\rho$ on the theoretical construction. If either the energy density or the closely related shape function is known, then the Einstein field equations do not ordinarily yield a finite redshift function. If, however, the wormhole admits a one-parameter group of conformal motions, then both the redshift and shape functions exist provided that $\omega < -1$. In a cosmological setting, the equation of state $p = \omega\rho$, $\omega < -1$, is associated with phantom dark energy, which is known to support traversable wormholes.

Keywords: Wormholes, Barotropic equation of state, Conformal symmetry

1 Introduction

Wormholes are tunnel-like structures connecting different universes or widely separated regions of our own Universe. That wormholes could be actual physical objects was first proposed by Morris and Thorne [1], who assumed that the wormhole spacetime can be

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described by the following static spherically symmetric line element:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{1}$$

The function $\Phi = \Phi(r)$ is called the *redshift function*, which must be everywhere finite to prevent an event horizon. The function $b = b(r)$ is called the *shape function* because it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram. The spherical surface $r = r_0$ is the *throat* of the wormhole and must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$, and $b'(r_0) < 1$, usually referred to as the *flare-out condition*. This condition refers to the flaring out of the embedding diagram pictured in Ref. [1]. The flare-out condition can only be met by violating the null energy condition.

Using units in which $c = G = 1$, the Einstein field equations in the orthonormal frame, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, yield the following simple interpretation for the components of the stress-energy tensor: $T_{\hat{t}\hat{t}} = \rho(r)$, the energy density, $T_{\hat{r}\hat{r}} = p_r$, the radial pressure, and $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p_t$, the lateral pressure. For the theoretical construction of the wormhole, Morris and Thorne then proposed the following strategy: retain complete control over the geometry by specifying the functions $b(r)$ and $\Phi(r)$ to obtain the desired properties. It is then up to the engineering team to search for or to manufacture the materials or fields that yield the required stress-energy tensor.

Researchers have tried various strategies for meeting this goal, some of which are discussed in the next section. These include the main subjects of this paper, the introduction of an appropriate equation of state and the geometric assumption of conformal symmetry, which leads to motions for which the metric tensor is invariant up to a scale factor.

2 Some stategies

To describe the various strategies, we must first list the Einstein field equations:

$$\rho(r) = \frac{b'}{8\pi r^2}, \tag{2}$$

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{3}$$

$$p_t(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[\Phi'' - \frac{b'r - b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r-b)}\right]. \tag{4}$$

Eq. (4) can actually be obtained from the conservation of the stress-energy tensor, i.e., ${T^{\mu\nu}}_{;\nu} = 0$; so only two of Eqs. (2)-(4) are independent. As a result, one can simply write

$$b' = 8\pi\rho r^2 \tag{5}$$

and

$$\Phi' = \frac{8\pi p_r r^3 + b}{2r(r-b)}. \tag{6}$$

Next, suppose we adopt the linear barotropic equation of state $p = \omega\rho$, which has been used in various cosmological settings. Since we are now dealing with wormholes, we will adopt the form

$$p_r = \omega\rho. \tag{7}$$

So if $\rho(r)$, or equivalently, $b(r)$ is assigned, one could conceivably obtain $p_r(r)$ and $\Phi(r)$ and hence a complete description of the wormhole geometry.

The energy density may also be known for physical reasons. One possibility is the Navarro-Frenk-White density profile in the dark-matter halo [2, 3]:

$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2}. \tag{8}$$

Here r_s is the characteristic scale radius and ρ_s the corresponding density.

Another possible physical reason for having $\rho(r)$ is noncommutative geometry, which replaces point particles by smeared objects in order to eliminate some of the divergences that normally appear in general relativity. The smearing effect is modeled by the use of either the Gaussian curve of minimal length $\sqrt{\theta}$ to represent the energy density [4, 5, 6, 8, 9]

$$\rho(r) = \frac{M}{(4\pi\theta)^{3/2}} e^{-r^2/4\theta} \tag{9}$$

or by the Lorentzian curve [10]

$$\rho(r) = \frac{M\sqrt{\theta}}{\pi^2(r^2 + \theta)^2}. \tag{10}$$

Here the mass M, instead of being perfectly localized, is diffused throughout the region due to the uncertainty. To clarify this statement, observe that the mass M_θ inside a sphere of radius r is

$$M_\theta = \int_{r_0}^{r} \rho(r')4\pi(r')^2 dr' = \frac{2M}{\pi}\left(\tan^{-1}\frac{r}{\sqrt{\theta}} - \frac{r\sqrt{\theta}}{r^2 + \theta}\right);$$

thus $M_\theta \to M$ as $\theta \to 0$. (In particular, when viewed from a distance, $M_\theta = M$.)

For all these cases, $b(r)$ can be computed from Eq. (5). So Eq. (7) then yields

$$\frac{1}{8\pi}\frac{b'}{r^2} = \frac{1}{\omega}\left(\frac{1}{8\pi}\right)\left[-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right]$$

and

$$\Phi' = \frac{\omega r b' + b}{2r^2\left(1 - \frac{b}{r}\right)}. \tag{11}$$

Recalling that $r = r_0$ at the throat, Φ' and hence Φ are not likely to be defined at $r = r_0$, thereby yielding an event horizon for negative ω, as, for example, in Ref. [11]. A proper choice of $b(r)$ can avoid this problem: $b(r) = r_0(r/r_0)^{-1/\omega}$ leads to $\Phi' \equiv 0$, thereby avoiding an event horizon [12].

A better way is to obtain $\Phi = \Phi(r)$ independently of the field equations. Thus Ref. [13] relies on both dark-matter and dark-energy models to obtain $\Phi(r)$.

In this paper we propose another method, the assumption that the wormhole admits a one-parameter group of conformal motions, discussed next.

3 Conformal Killing vectors

As noted in the Introduction, we assume that our spacetime admits a one-parameter group of conformal motions, which are motions along which the metric tensor of a spacetime remains invariant up to a scale factor. This is equivalent to the existence of conformal Killing vectors such that

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\eta\nu}\, \xi^\eta{}_{;\mu} + g_{\mu\eta}\, \xi^\eta{}_{;\nu} = \psi(r)\, g_{\mu\nu}, \tag{12}$$

where the left-hand side is the Lie derivative of the metric tensor and $\psi(r)$ is the conformal factor. (For further discussion, see [14, 15].) The vector ξ generates the conformal symmetry and the metric tensor $g_{\mu\nu}$ is conformally mapped into itself along ξ. This type of symmetry has been used to great advantage in describing relativistic stellar-type objects [16, 17]. Besides leading to new solutions, the conformal symmetries have led to new geometric and kinematical insights [18, 19, 20, 21].

Exact solutions of traversable wormholes admitting conformal motions have also been found, given a noncommutative-geometry background [22]. Two earlier studies assumed a *non-static* conformal symmetry [15, 23].

To discuss conformal symmetry, it is convenient to use the following form of the metric [22]:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{13}$$

Now the Einstein field equations take on the following form:

$$e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = 8\pi\rho, \tag{14}$$

$$e^{-\lambda}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] - \frac{1}{r^2} = 8\pi p_r, \tag{15}$$

and

$$\frac{1}{2}e^{-\lambda}\left[\frac{1}{2}(\nu')^2 + \nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\right] = 8\pi p_t. \tag{16}$$

Next, we turn our attention to the assumption of conformal symmetry in Eq. (12). Here we follow Herrera and Ponce de León [16] and restrict the vector field by requiring that $\xi^\alpha U_\alpha = 0$, where U_α is the four-velocity of the perfect fluid distribution. The assumption of spherical symmetry then implies that $\xi^0 = \xi^2 = \xi^3 = 0$ [16]. Eq. (12) now yields the following results:

$$\xi^1\nu' = \psi, \tag{17}$$

$$\xi^1 = \frac{\psi r}{2}, \tag{18}$$

and

$$\xi^1\lambda' + 2\,\xi^1{}_{,1} = \psi. \tag{19}$$

From these equations we obtain

$$e^\nu = C_1 r^2 \tag{20}$$

and

$$e^{\lambda} = \left(\frac{a}{\psi}\right)^2, \tag{21}$$

where C_1 and a are integration constants. In order to make use of Eqs. (20) and (21), we rewrite Eqs. (14)-(16) as follows:

$$\frac{1}{r^2}\left(1 - \frac{\psi^2}{a^2}\right) - \frac{2\psi\psi'}{a^2 r} = 8\pi\rho, \tag{22}$$

$$\frac{1}{r^2}\left(\frac{3\psi^2}{a^2} - 1\right) = 8\pi p_r, \tag{23}$$

and

$$\frac{\psi^2}{a^2 r^2} + \frac{2\psi\psi'}{a^2 r} = 8\pi p_t. \tag{24}$$

4 Wormhole structure

Returning to the equation of state (7), $p_r = \omega\rho$, Eqs. (22) and (23) yield (after some simplification)

$$2r\omega\Psi\Psi' + (\omega + 3)\Psi^2 = a^2(\omega + 1). \tag{25}$$

Noting that $2\psi\psi' = (\psi^2)'$, the equation is linear in ψ^2 and can be readily solved to yield

$$\Psi^2(r) = \frac{1}{\omega + 3}[a^2(\omega + 1) + (\omega + 3)cr^{-(\omega+3)/\omega}], \tag{26}$$

where c is an integration constant. Comparing line elements (1) and (13), we have in view of Eq. (21),

$$b(r) = r(1 - e^{-\lambda}) = r\left(1 - \frac{\psi^2}{a^2}\right), \tag{27}$$

which yields the following class of shape functions:

$$b(r) = r\left(\frac{2}{\omega + 3} - \frac{c}{a^2}r^{-(\omega+3)/\omega}\right). \tag{28}$$

The requirement $b(r_0) = r_0$ can be used to determine the integration constant. In particular, from Eq. (28),

$$\frac{2}{\omega + 3} - \frac{c}{a^2}r_0^{-(\omega+3)/\omega} = 1$$

and

$$c = -\frac{\omega + 1}{\omega + 3}a^2 r_0^{(\omega+3)/\omega}.$$

So Eq. (28) becomes

$$b(r) = r\left(\frac{2}{\omega + 3} + \frac{\omega + 1}{\omega + 3}r_0^{(\omega+3)/\omega}r^{-(\omega+3)/\omega}\right). \tag{29}$$

The next step is to check the flare-out condition $b'(r_0) < 1$. From Eq. (29),

$$b'(r) = \frac{2}{\omega+3} + \frac{\omega+1}{\omega+3} r_0^{(\omega+3)/\omega} r^{-(\omega+3)/\omega} + r\left[\frac{\omega+1}{\omega+3} r_0^{(\omega+3)/\omega}\left(-\frac{\omega+3}{\omega} r^{-(\omega+3)/\omega-1}\right)\right]. \quad (30)$$

After substituting r_0 for r and simplifying, we obtain

$$b'(r_0) = -\frac{1}{\omega} < 1 \quad (31)$$

since $\omega < -1$. So the flare-out condition is met.

This result shows that a wormhole solution requires a phantom-energy background, i.e., $\omega < -1$, which is consistent with earlier studies [11, 24, 25, 26]. (The reason is that whenever $\omega < -1$ in the equation of state $p = \omega\rho$, the null energy condition is violated.)

To obtain the redshift function from Eq. (20), we need to determine the integration constant C_1, discussed next.

5 Junction to an external vacuum solution

Eq. (20) implies that the wormhole spacetime is not asymptotically flat. So the wormhole material must be cut off at some $r = r_1$ and joined to the exterior Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (32)$$

Here

$$M = \frac{1}{2}b(r_1) = \frac{1}{2}r_1\left(\frac{2}{\omega+3} + \frac{\omega+1}{\omega+3} r_0^{(\omega+3)/\omega} r_1^{-(\omega+3)/\omega}\right). \quad (33)$$

So for $e^\nu = C_1 r^2$ from Eq. (20), we have $C_1 r_1^2 = 1 - 2M/r_1$ and the integration constant becomes

$$C_1 = \frac{1}{r_1^2}\left(1 - \frac{2M}{r_1}\right), \quad (34)$$

where M is given in Eq. (33). This completes the wormhole solution.

6 Conclusion

After establishing that the adoption of the linear barotropic equation of state $p_r = \omega\rho$ is usually insufficient for obtaining a finite redshift function from the Einstein field equations, it is shown in this paper that such a redshift function can be obtained by assuming that the wormhole admits a one-parameter group of conformal motions. The solution requires a phantom-energy background, however, a conclusion that is consistent with earlier studies. The resulting wormhole is not asymptotically flat and must be joined to an external Schwarzschild spacetime.

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